Significance

Does this value of acceleration seem astoundingly small? If they start from rest, then they would accelerate directly toward each other, “colliding” at their center of mass. Let’s estimate the time for this to happen. The initial acceleration is $\sim 10^{-13}\ \text{m/s}^2$, so using $v = at$, we see that it would take $\sim 10^{13}\ \text{s}$ for each galaxy to reach a speed of 1.0 m/s, and they would be only $\sim 0.5 \times 10^{13}\ \text{m}$ closer. That is nine orders of magnitude smaller than the initial distance between them. In reality, such motions are rarely simple. These two galaxies, along with about 50 other smaller galaxies, are all gravitationally bound into our local cluster. Our local cluster is gravitationally bound to other clusters in what is called a supercluster. All of this is part of the great cosmic dance that results from gravitation, as shown in **Figure 13.6**.

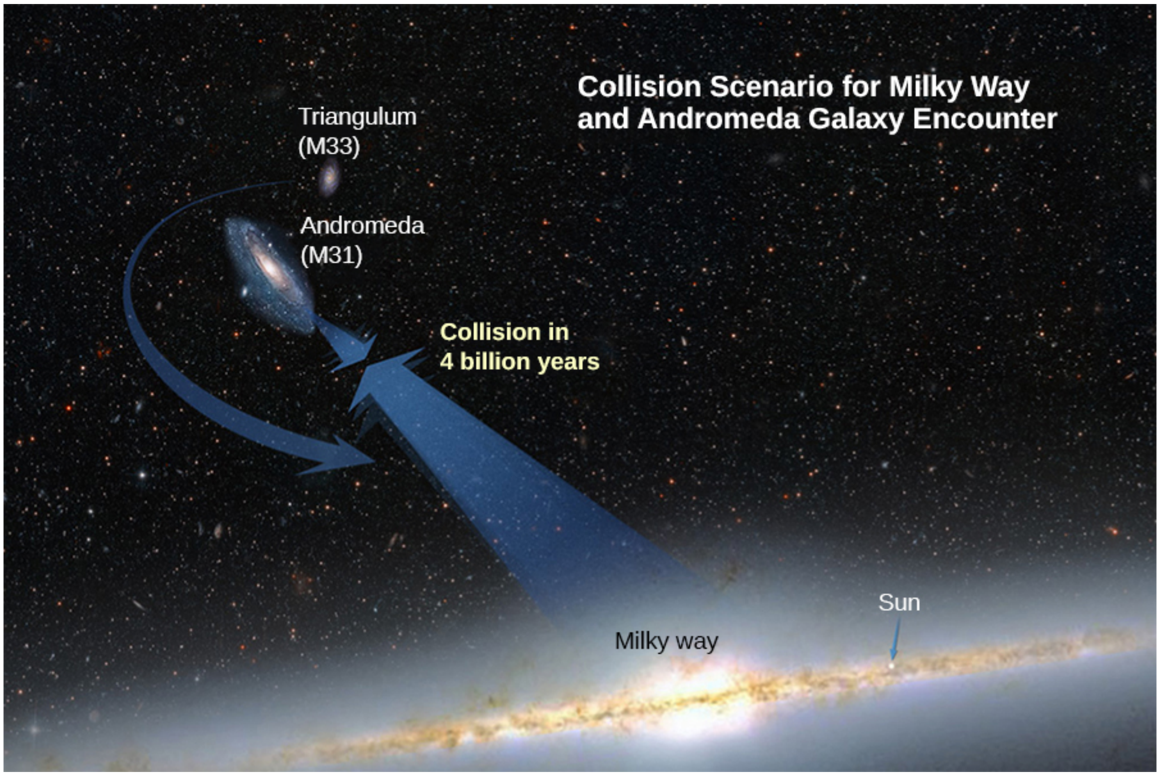

Figure 13.6 Based on the results of this example, plus what astronomers have observed elsewhere in the Universe, our galaxy will collide with the Andromeda Galaxy in about 4 billion years. (credit: modification of work by NASA; ESA; A. Feild and R. van der Marel, STScI)

13.2 | Gravitation Near Earth's Surface

Learning Objectives

By the end of this section, you will be able to:

- Explain the connection between the constants G and g
- Determine the mass of an astronomical body from free-fall acceleration at its surface
- Describe how the value of g varies due to location and Earth’s rotation

In this section, we observe how Newton’s law of gravitation applies at the surface of a planet and how it connects with what we learned earlier about free fall. We also examine the gravitational effects within spherical bodies.

Weight

Recall that the acceleration of a free-falling object near Earth's surface is approximately $g = 9.80\ \text{m/s}^2$. The force causing this acceleration is called the weight of the object, and from Newton's second law, it has the value *mg*. This weight is present regardless of whether the object is in free fall. We now know that this force is the gravitational force between the object and Earth. If we substitute *mg* for the magnitude of $\vec{\mathbf{F}}_{12}$ in Newton's law of universal gravitation, *m* for m_1, and M_E for m_2, we obtain the scalar equation

$$mg = G\frac{mM_\text{E}}{r^2}$$

where *r* is the distance between the centers of mass of the object and Earth. The average radius of Earth is about 6370 km. Hence, for objects within a few kilometers of Earth's surface, we can take $r = R_\text{E}$ (**Figure 13.7**). The mass *m* of the object cancels, leaving

$$g = G\frac{M_\text{E}}{r^2}. \tag{13.2}$$

This explains why all masses free fall with the same acceleration. We have ignored the fact that Earth also accelerates toward the falling object, but that is acceptable as long as the mass of Earth is much larger than that of the object.

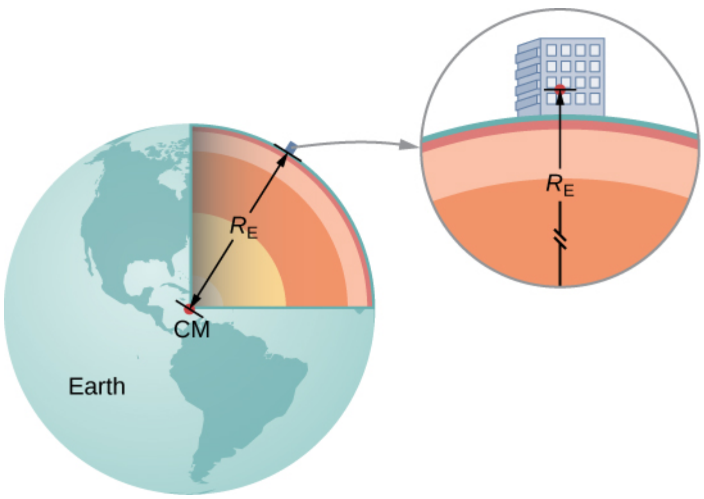

Figure 13.7 We can take the distance between the centers of mass of Earth and an object on its surface to be the radius of Earth, provided that its size is much less than the radius of Earth.

Example 13.3

Masses of Earth and Moon

Have you ever wondered how we know the mass of Earth? We certainly can't place it on a scale. The values of *g* and the radius of Earth were measured with reasonable accuracy centuries ago.

a. Use the standard values of *g*, R_E, and **Equation 13.2** to find the mass of Earth.

b. Estimate the value of *g* on the Moon. Use the fact that the Moon has a radius of about 1700 km (a value of this accuracy was determined many centuries ago) and assume it has the same average density as Earth, $5500\ \text{kg/m}^3$.

Strategy

With the known values of g and R_E, we can use **Equation 13.2** to find M_E. For the Moon, we use the assumption of equal average density to determine the mass from a ratio of the volumes of Earth and the Moon.

Solution

a. Rearranging **Equation 13.2**, we have

$$M_E = \frac{gR_E^2}{G} = \frac{9.80\ \text{m/s}^2 (6.37 \times 10^6\ \text{m})^2}{6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2} = 5.95 \times 10^{24}\ \text{kg}.$$

b. The volume of a sphere is proportional to the radius cubed, so a simple ratio gives us

$$\frac{M_M}{M_E} = \frac{R_M^3}{R_E^3} \rightarrow M_M = \left(\frac{(1.7 \times 10^6\ \text{m})^3}{(6.37 \times 10^6\ \text{m})^3}\right)(5.95 \times 10^{24}\ \text{kg}) = 1.1 \times 10^{23}\ \text{kg}.$$

We now use **Equation 13.2**.

$$g_M = G\frac{M_M}{r_M^2} = (6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2)\frac{(1.1 \times 10^{23}\ \text{kg})}{(1.7 \times 10^6\ \text{m})^2} = 2.5\ \text{m/s}^2$$

Significance

As soon as Cavendish determined the value of G in 1798, the mass of Earth could be calculated. (In fact, that was the ultimate purpose of Cavendish's experiment in the first place.) The value we calculated for g of the Moon is incorrect. The average density of the Moon is actually only $3340\ \text{kg/m}^3$ and $g = 1.6\ \text{m/s}^2$ at the surface. Newton attempted to measure the mass of the Moon by comparing the effect of the Sun on Earth's ocean tides compared to that of the Moon. His value was a factor of two too small. The most accurate values for g and the mass of the Moon come from tracking the motion of spacecraft that have orbited the Moon. But the mass of the Moon can actually be determined accurately without going to the Moon. Earth and the Moon orbit about a common center of mass, and careful astronomical measurements can determine that location. The ratio of the Moon's mass to Earth's is the ratio of [the distance from the common center of mass to the Moon's center] to [the distance from the common center of mass to Earth's center].

Later in this chapter, we will see that the mass of other astronomical bodies also can be determined by the period of small satellites orbiting them. But until Cavendish determined the value of G, the masses of all these bodies were unknown.

Example 13.4

Gravity above Earth's Surface

What is the value of g 400 km above Earth's surface, where the International Space Station is in orbit?

Strategy

Using the value of M_E and noting the radius is $r = R_E + 400\ \text{km}$, we use **Equation 13.2** to find g.

From **Equation 13.2** we have

$$g = G\frac{M_E}{r^2} = 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2 \frac{5.96 \times 10^{24}\ \text{kg}}{(6.37 \times 10^6 + 400 \times 10^3\ \text{m})^2} = 8.67\ \text{m/s}^2.$$

Significance

We often see video of astronauts in space stations, apparently weightless. But clearly, the force of gravity is acting on them. Comparing the value of g we just calculated to that on Earth $(9.80\ \text{m/s}^2)$, we see that the astronauts in the International Space Station still have 88% of their weight. They only appear to be weightless because they are in free fall. We will come back to this in **Satellite Orbits and Energy**.

13.2 Check Your Understanding How does your weight at the top of a tall building compare with that on the first floor? Do you think engineers need to take into account the change in the value of g when designing structural support for a very tall building?

The Gravitational Field

Equation 13.2 is a scalar equation, giving the magnitude of the gravitational acceleration as a function of the distance from the center of the mass that causes the acceleration. But we could have retained the vector form for the force of gravity in Equation 13.1, and written the acceleration in vector form as

$$\vec{\mathbf{g}} = G\frac{M}{r^2}\hat{\mathbf{r}}.$$

We identify the vector field represented by $\vec{\mathbf{g}}$ as the **gravitational field** caused by mass M. We can picture the field as shown Figure 13.8. The lines are directed radially inward and are symmetrically distributed about the mass.

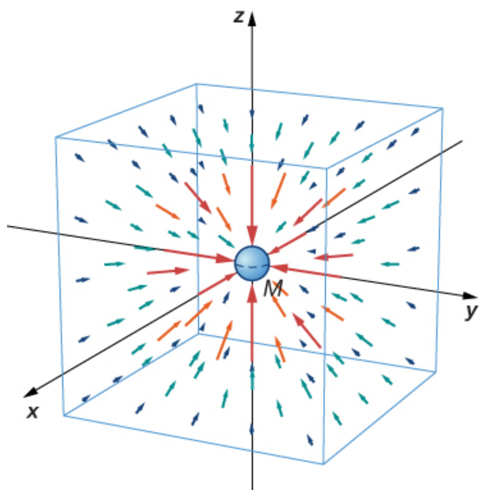

Figure 13.8 A three-dimensional representation of the gravitational field created by mass M. Note that the lines are uniformly distributed in all directions. (The box has been added only to aid in visualization.)

As is true for any vector field, the direction of $\vec{\mathbf{g}}$ is parallel to the field lines at any point. The strength of $\vec{\mathbf{g}}$ at any point is inversely proportional to the line spacing. Another way to state this is that the magnitude of the field in any region is proportional to the number of lines that pass through a unit surface area, effectively a density of lines. Since the lines are equally spaced in all directions, the number of lines per unit surface area at a distance r from the mass is the total number of lines divided by the surface area of a sphere of radius r, which is proportional to r^2. Hence, this picture perfectly represents the inverse square law, in addition to indicating the direction of the field. In the field picture, we say that a mass m interacts with the gravitational field of mass M. We will use the concept of fields to great advantage in the later chapters on electromagnetism.

Apparent Weight: Accounting for Earth's Rotation

As we saw in Applications of Newton's Laws, objects moving at constant speed in a circle have a centripetal acceleration directed toward the center of the circle, which means that there must be a net force directed toward the center of that circle. Since all objects on the surface of Earth move through a circle every 24 hours, there must be a net centripetal force on each object directed toward the center of that circle.

Let's first consider an object of mass m located at the equator, suspended from a scale (Figure 13.9). The scale exerts an upward force $\vec{\mathbf{F}}_s$ away from Earth's center. This is the reading on the scale, and hence it is the **apparent weight** of the object. The weight (mg) points toward Earth's center. If Earth were not rotating, the acceleration would be zero and, consequently, the net force would be zero, resulting in $F_s = mg$. This would be the true reading of the weight.

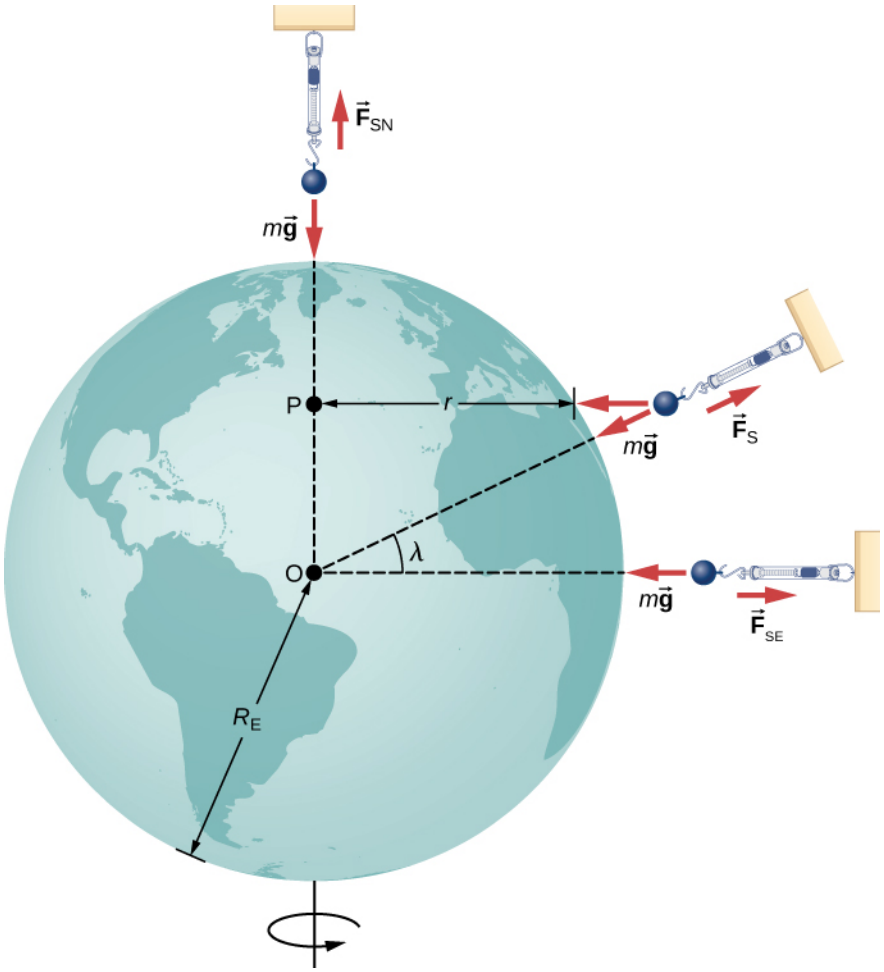

Figure 13.9 For a person standing at the equator, the centripetal acceleration (a_c) is in the same direction as the force of gravity. At latitude λ, the angle the between a_c and the force of gravity is λ and the magnitude of a_c decreases with $\cos\lambda$.

With rotation, the sum of these forces must provide the centripetal acceleration, a_c. Using Newton's second law, we have

$$\sum F = F_s - mg = ma_c \quad \text{where} \quad a_c = -\frac{v^2}{r}. \tag{13.3}$$

Note that a_c points in the same direction as the weight; hence, it is negative. The tangential speed v is the speed at the equator and r is R_E. We can calculate the speed simply by noting that objects on the equator travel the circumference of Earth in 24 hours. Instead, let's use the alternative expression for a_c from **Motion in Two and Three Dimensions**. Recall that the tangential speed is related to the angular speed (ω) by $v = r\omega$. Hence, we have $a_c = -r\omega^2$. By rearranging **Equation 13.3** and substituting $r = R_E$, the apparent weight at the equator is

$$F_s = m\left(g - R_E\omega^2\right).$$

The angular speed of Earth everywhere is

$$\omega = \frac{2\pi \text{ rad}}{24 \text{ hr} \times 3600 \text{ s/hr}} = 7.27 \times 10^{-5} \text{ rad/s}.$$

Substituting for the values or R_E and ω, we have $R_E\omega^2 = 0.0337 \text{ m/s}^2$. This is only 0.34% of the value of gravity, so it is clearly a small correction.

Example 13.5

Zero Apparent Weight

How fast would Earth need to spin for those at the equator to have zero apparent weight? How long would the length of the day be?

Strategy

Using **Equation 13.3**, we can set the apparent weight (F_s) to zero and determine the centripetal acceleration required. From that, we can find the speed at the equator. The length of day is the time required for one complete rotation.

Solution

From **Equation 13.2**, we have $\sum F = F_s - mg = ma_c$, so setting $F_s = 0$, we get $g = a_c$. Using the expression for a_c, substituting for Earth's radius and the standard value of gravity, we get

$$a_c = \frac{v^2}{r} = g$$
$$v = \sqrt{gr} = \sqrt{(9.80 \text{ m/s}^2)(6.37 \times 10^6 \text{ m})} = 7.91 \times 10^3 \text{ m/s}.$$

The period *T* is the time for one complete rotation. Therefore, the tangential speed is the circumference divided by *T*, so we have

$$v = \frac{2\pi r}{T}$$
$$T = \frac{2\pi r}{v} = \frac{2\pi(6.37 \times 10^6 \text{ m})}{7.91 \times 10^3 \text{ m/s}} = 5.06 \times 10^3 \text{ s}.$$

This is about 84 minutes.

Significance

We will see later in this chapter that this speed and length of day would also be the orbital speed and period of a satellite in orbit at Earth's surface. While such an orbit would not be possible near Earth's surface due to air resistance, it certainly is possible only a few hundred miles above Earth.

Results Away from the Equator

At the poles, $a_c \to 0$ and $F_s = mg$, just as is the case without rotation. At any other latitude λ, the situation is more complicated. The centripetal acceleration is directed toward point *P* in the figure, and the radius becomes $r = R_E \cos\lambda$. The *vector* sum of the weight and $\vec{\mathbf{F}}_s$ must point toward point *P*, hence $\vec{\mathbf{F}}_s$ no longer points away from the center of Earth. (The difference is small and exaggerated in the figure.) A plumb bob will always point along this deviated direction. All buildings are built aligned along this deviated direction, not along a radius through the center of Earth. For the tallest buildings, this represents a deviation of a few feet at the top.

It is also worth noting that Earth is not a perfect sphere. The interior is partially liquid, and this enhances Earth bulging at the equator due to its rotation. The radius of Earth is about 30 km greater at the equator compared to the poles. It is left as an exercise to compare the strength of gravity at the poles to that at the equator using **Equation 13.2**. The difference is comparable to the difference due to rotation and is in the same direction. Apparently, you really can lose "weight" by moving to the tropics.

Gravity Away from the Surface

Earlier we stated without proof that the law of gravitation applies to spherically symmetrical objects, where the mass of each body acts as if it were at the center of the body. Since **Equation 13.2** is derived from **Equation 13.1**, it is also valid for symmetrical mass distributions, but both equations are valid only for values of $r \geq R_E$. As we saw in **Example 13.4**, at 400 km above Earth's surface, where the International Space Station orbits, the value of g is $8.67\ \text{m/s}^2$. (We will see later that this is also the centripetal acceleration of the ISS.)

For $r < R_E$, **Equation 13.1** and **Equation 13.2** are not valid. However, we can determine g for these cases using a principle that comes from Gauss's law, which is a powerful mathematical tool that we study in more detail later in the course. A consequence of Gauss's law, applied to gravitation, is that only the mass *within* r contributes to the gravitational force. Also, that mass, just as before, can be considered to be located at the center. The gravitational effect of the mass *outside* r has zero net effect.

Two very interesting special cases occur. For a spherical planet with constant density, the mass within r is the density times the volume within r. This mass can be considered located at the center. Replacing M_E with only the mass within r, $M = \rho \times (\text{volume of a sphere})$, and R_E with r, **Equation 13.2** becomes

$$g = G\frac{M_E}{R_E^2} = G\frac{\rho\left(4/3\pi r^3\right)}{r^2} = \frac{4}{3}G\rho\pi r.$$

The value of g, and hence your weight, decreases linearly as you descend down a hole to the center of the spherical planet. At the center, you are weightless, as the mass of the planet pulls equally in all directions. Actually, Earth's density is not constant, nor is Earth solid throughout. **Figure 13.10** shows the profile of g if Earth had constant density and the more likely profile based upon estimates of density derived from seismic data.

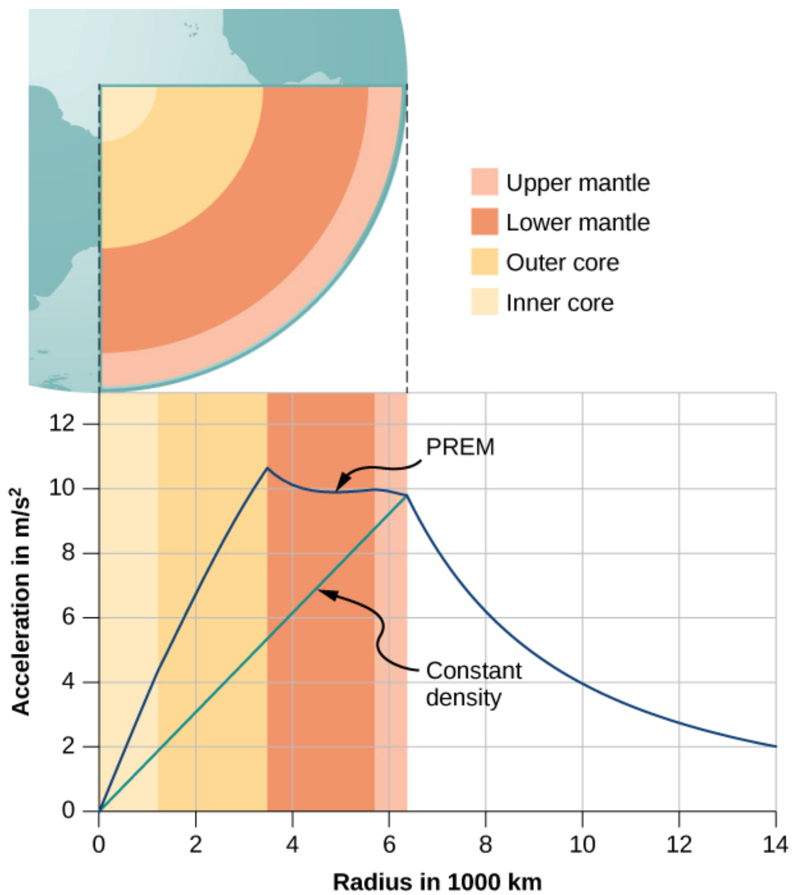

Figure 13.10 For $r < R_E$, the value of g for the case of constant density is the straight green line. The blue line from the PREM (Preliminary Reference Earth Model) is probably closer to the actual profile for g.

The second interesting case concerns living on a spherical shell planet. This scenario has been proposed in many science fiction stories. Ignoring significant engineering issues, the shell could be constructed with a desired radius and total mass, such that g at the surface is the same as Earth's. Can you guess what happens once you descend in an elevator to the inside of the shell, where there is no mass between you and the center? What benefits would this provide for traveling great distances from one point on the sphere to another? And finally, what effect would there be if the planet was spinning?

13.3 | Gravitational Potential Energy and Total Energy

Learning Objectives

By the end of this section, you will be able to:

- Determine changes in gravitational potential energy over great distances
- Apply conservation of energy to determine escape velocity
- Determine whether astronomical bodies are gravitationally bound

We studied gravitational potential energy in **Potential Energy and Conservation of Energy**, where the value of g remained constant. We now develop an expression that works over distances such that g is not constant. This is necessary to correctly calculate the energy needed to place satellites in orbit or to send them on missions in space.

Gravitational Potential Energy beyond Earth

We defined work and potential energy in **Work and Kinetic Energy** and **Potential Energy and Conservation of Energy**. The usefulness of those definitions is the ease with which we can solve many problems using conservation of energy. Potential energy is particularly useful for forces that change with position, as the gravitational force does over large distances. In **Potential Energy and Conservation of Energy**, we showed that the change in gravitational potential energy near Earth's surface is $\Delta U = mg(y_2 - y_1)$. This works very well if g does not change significantly between y_1 and y_2. We return to the definition of work and potential energy to derive an expression that is correct over larger distances.

Recall that work (W) is the integral of the dot product between force and distance. Essentially, it is the product of the component of a force along a displacement times that displacement. We define ΔU as the *negative* of the work done by the force we associate with the potential energy. For clarity, we derive an expression for moving a mass m from distance r_1 from the center of Earth to distance r_2. However, the result can easily be generalized to any two objects changing their separation from one value to another.

Consider **Figure 13.11**, in which we take m from a distance r_1 from Earth's center to a distance that is r_2 from the center. Gravity is a conservative force (its magnitude and direction are functions of location only), so we can take any path we wish, and the result for the calculation of work is the same. We take the path shown, as it greatly simplifies the integration. We first move *radially* outward from distance r_1 to distance r_2, and then move along the arc of a circle until we reach the final position. During the radial portion, $\vec{\mathbf{F}}$ is opposite to the direction we travel along $d\vec{\mathbf{r}}$, so $E = K_1 + U_1 = K_2 + U_2$. Along the arc, $\vec{\mathbf{F}}$ is perpendicular to $d\vec{\mathbf{r}}$, so $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$. No work is done as we move along the arc. Using the expression for the gravitational force and noting the values for $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ along the two segments of our path, we have

$$\Delta U = -\int_{r_1}^{r_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = GM_{\text{E}}m\int_{r_1}^{r_2}\frac{dr}{r^2} = GM_{\text{E}}m\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Since $\Delta U = U_2 - U_1$, we can adopt a simple expression for U:

$$U = -\frac{GM_{\text{E}}m}{r}. \tag{13.4}$$